# Fundamental Properties of Homogeneous Multifractals

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フラクタルの統計法則

# Fundamental Properties of Homogeneous Multifractals[1]

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It is proved from the requirement of scale-similarity of multifractals that the probability of spatial distribution of a certain measure supported by a multifractal, which may be called intrinsic probability, is uniquely determined for scale ratio tending to zero if the f-$\alpha$ spectrum of the multifractal is given. As a corollary, it is proved that there exists no nonlinear transformation of multifractals. Also, it is derived that intrinsic probabilities of many multifractals including multi-nomial generalized Cantor sets can be determined by the knowledge of intermittency exponents $\mu(q)$ (and then generalized dimensions $D(q)$) limited for q = nonnegative integers only.

[1]The content was spoken in the Symp. on "Generation and Statistical Law of Turbulence " at Res. Inst. Math. Sci., Kyoto Univ. on Jan. 21-23, 1992, except for the 1st corollary.

In the previous paper[1], it was clarified that there are generalized dimensions D(q), f-$\alpha$ spectrum f($\alpha$), intermittency exponents $\mu$(q), and intrinsic probability p(y; r/l) (for an arbitrary scale ratio r/l) associated with, and to characterize, every isotropic homogeneous multifractal; these quantities are equivalent to each other; and also D(q) and $\mu$(q) are continuous and differentiable in q, if D(q) is defined in the sense of Hentschel and Procaccia[2].

Here we resume the proof of the uniqueness for a given f($\alpha$) of p(y; r/l) that is the probability density of spatial distribution of a certain measure supported by the multifractal for a scale ratio r/l tending to zero, by a new argument using scale-similarity of a multifractal.

Returning to Hentschel and Procaccia's formula[2], we have

$$\sum_i (p_i^{(r/L)})^q = C_q(r/L)^{(q-1)D(q)} \tag{1}$$

as $r/L \rightarrow 0$. L is a main scale, $C_q$ is proper proportioinal constants, and $p_i^{(r/L)}$ is the normalized measure of the ith subbox of scale r in the box of scale L. The sum $\sum_i$ denotes summation over all subboxes of scale r, except for the ones with $p_i^{(r/L)} = 0$. Here we consider much finer subboxes of scale s. Then, of course, we should have a similar formula,

$$\sum_k (p_k^{(s/L)})^q = C_q(s/L)^{(q-1)D(q)} \quad . \tag{2}$$

Here we can find $(r/s)^d$ subboxes of scale s in each subbox of scale r. The measure of the jth subbox of scale s in the ith subbox of scale r may be expressed as $p_{ji}^{(s/r)}p_i^{(r/L)}$. Of course, we have $\sum_j p_{ji}^{(s/r)} = 1$. Then, if s/r goes to zero, we should have

$$\sum_j (p_{ji}^{(s/r)})^q = C_q(s/r)^{(q-1)D(q)} \tag{3}$$

in each subbox of scale r (irrespectively of i), so long as the multifractal nature of the total domain is the same as that of a partial domain, as is illustrated in Fig. 1. Consider the measure of the kth subbox of scale s is that of the jth subbox of scale s in the ith subbox of scale r, so that

$$p_k^{(s/L)} = p_{ji}^{(s/r)} p_i^{(r/L)}. \tag{4}$$

Thus, we have

$$\Sigma_k (p_k^{(s/L)})^q = \Sigma_{i,j} [p_{ji}^{(s/r)} p_i^{(r/L)}]^q = \Sigma_i (p_i^{(r/L)})^q \ \Sigma_j (p_{ji}^{(s/r)})^q. \tag{5}$$

Hence, we have from (1), (2) and (3) that

$$C_q = C_q^2 . \tag{6}$$

Therefore, $C_q = 1$ is the only meaningful case that we can have for all q. The above argument may be a little more relaxed for the most general case where $C_q$ depends weakly on r/L, such as including the the the ln(r/L) factor. However, such a case should be prohibited that $C_q$ is a power function of r/L, because it violates the definition of $D_q$. If we start from the premise of $C_q(r/L)$, we have

$$C_q(s/L) = C_q(s/r) C_q(r/L) \tag{7}$$

in place of (6). Then, the only possible solution is

$$C_q(r/L) = (r/L)^{\nu_q}, \qquad \nu_q : \text{const.} \tag{8}$$

But this is the prohibited case unless $\nu_q = 0$.

Thus, if the left-hand side of (1) is replaced by the often-used heuristic expression in terms of $f(\alpha)$, we should accept the exact equality:

$$\int \rho(\alpha)(r/L)^{-f(\alpha)+q\alpha} d\alpha = (r/L)^{(q-1)D(q)}; \tag{9}$$

$\rho(\alpha)$ denotes a weight in the integration. Now we prove that $\rho(\alpha)$ can take the only one form. The steepest descent method allows us to write (9) as

$$\rho(\alpha_1)(r/L)^{q\alpha_1 - f(\alpha_1)} \int (r/L)^{-f''(\alpha)(\alpha-\alpha_1)^2/2} d\alpha = (r/L)^{(q-1)D_q} \tag{10}$$

under the condition of $q - f'(\alpha) = 0$ and $f''(\alpha) < 0$, and to give

$$\alpha_1 q - f(\alpha_1) = (q-1)D(q) \tag{11}$$

as well as

$$\rho(\alpha_1) = [f''(\alpha_1)\ln(r/L)/(2\pi)]^{1/2} \tag{12}$$

for every q in $(-\infty, \infty)$ and then for every $\alpha_1$ in its whole range, because

$\alpha_1$must change continuously depending on q by (11), if D(q) and $f(\alpha_1)$ are continuous functions. This concludes the proof. We note here that (12) is the lowest-order asymptotic formula to relate $\rho$ to f, but it is easy to obtain higher-order formulas including the correction terms of $O[|\ln(r/L)|^{-n}]$ (n = 1, 2,..). The special case with a one-point f-$\alpha$ spectrum cannot be treated by the above argument. The previous treatment[1)] contains exactly this case.

Thus, it is incorrect to presume $\rho(\alpha)$ in an arbitrary form, since it evidently destroys the equality of (10) and then (9) for all q or violates scale-similarity of multifractals. Physically, this means that the probability distribution of $\alpha$ in space for scale ratio r/L → 0 should be decided by the f-$\alpha$ spectrum alone, and never interfered by an extra independent factor. The present proof is less axiomatic but more illustrative than the previous one.

As a corollary, we can find the transormation rule of multifractals. Suppose two multifractals with $f_i(\alpha_i)$, $\rho_i(\alpha_i)$, and $D_i(q)$ (for i = 1, 2). Then we have

$$\int \rho_i(\alpha_i)(r/L)^{-f(\alpha_i)+q\alpha_i}d\alpha_i = (r/L)^{(q-1)D_i(q)}. \tag{13}$$

If $\alpha_2$ is related to $\alpha_1$ as $\alpha_2(\alpha_1)$, (13) for i = 2 is rewritten as

$$\int \rho_2(\alpha_2)\alpha_2'(\alpha_1)(r/L)^{-f(\alpha_2)+q\alpha_2}d\alpha_1 = (r/L)^{(q-1)D_2(q)}. \tag{14}$$

Since the right-hand side is the ensemble average of $(r/L)^{q\alpha_2-d}$ (d: spatial dimension), we must have just

$$\rho_2(\alpha_2)\alpha_2'(\alpha_1) = \rho_1(\alpha_1) \text{ and } f_2(\alpha_2(\alpha_1)) = f_1(\alpha_1). \tag{15}$$

As a result, we can produce a different multifractal with $f_1(\alpha_1)$ from a multifractal with $f_2(\alpha_2)$ and vice versa by way of (15), once a function $\alpha_2(\alpha_1)$ is given. How arbitrary is the functional form of $\alpha_2(\alpha_1)$? According to (12), we have

$$\rho_i(\alpha_i) = [f''_i(\alpha_i)\ln(r/L)/(2\pi)]^{1/2}. \tag{16}$$

On the other hand, we have

$$\partial^2/\partial\alpha_1^2\ f_2(\alpha_2(\alpha_1)) = f_2''(\alpha_2)(\alpha_2')^2 + f_2'(\alpha_2)\alpha_2'' \tag{17}$$

Thus, (15), (16) and (17) require

$$\alpha_2''(\alpha_1) = 0, \tag{18}$$

which means

$$\alpha_2(\alpha_1) = a\,\alpha_1 + b \quad (a, b : \text{const}) . \tag{19}$$

Namely, any nonlinear transformation of $\alpha$ is prohibited. It is easy to obtain the transformation rule of D(q) caused by (19) as

$$(q-1)D_2(q) = (aq-1)D_1(aq) + bq. \tag{20}$$

As another corollary, we can argue the moment problem of intrinsic probability. The intrinsic probability to characterize a multifractal may be written as[1)]

$$(r/l)^{\mu(q)} = \int_0^{(l/r)^d} y^q p(y; r/l)dy. \tag{21}$$

In this case, r/l does not necessarily tend to zero. Since (21) is a kind of the Mellin transform, we generally need the knowledge of $\mu(q)$ in the complex q-plane in order to determine the form of p. It is to be noted here that all formes for $\mu(q)$ do not necessarily give the intrinsic probability of a multifractal which is strongly conditioned to vanish towards y = 0 (rapidly) and for $y > (l/r)^d$; neither an exponential nor lognormal form as p is not exactly conditioned to do so. Many forms of p which can characterize multifractals were shown in Ref. 3, including binomial generalized Cantor sets; it is easy to extend the argument to multi-nomial Cantor sets. It is obvious that the characteristic functions $\phi(\theta; r/l)$ of these intrinsic probabilities have no essential singularity at $\theta = 0$; that is, $\phi$ can be expanded in a Taylor series. Since all the Taylor coefficients at the origin are given by all the nonnegative-integer-order moments of y, many intrinsic probabilities of multifractals can be determined by the limited knowledge of $\mu(q)$ only for q = 0, 1, 2, . . . In these cases, all other values of $\mu(q)$ and D(q) are redundunt. Correspondingly, the right branch of $f(\alpha)$ is redundunt because it is decided by D(q) for $q < 0$. Also, it is remarked that intrinsic probabilities of these multifractals are much less intermittent than the lognormal distribution that was mentioned by Orszag[4] as an example in which all the moments of nonnegative-integer-order cannot determine a unique probability. It is easy to see that the moments of y in generalized Cantor sets are within Carleman's criterion[4].

Finally, we note that the longitudinal velocity difference in isotropic turbulence is not supported by a multifractal in the present paradigm, because it is not exactly a measure in space. The statistical quality of it was discussed in Refs. 5 and 6.

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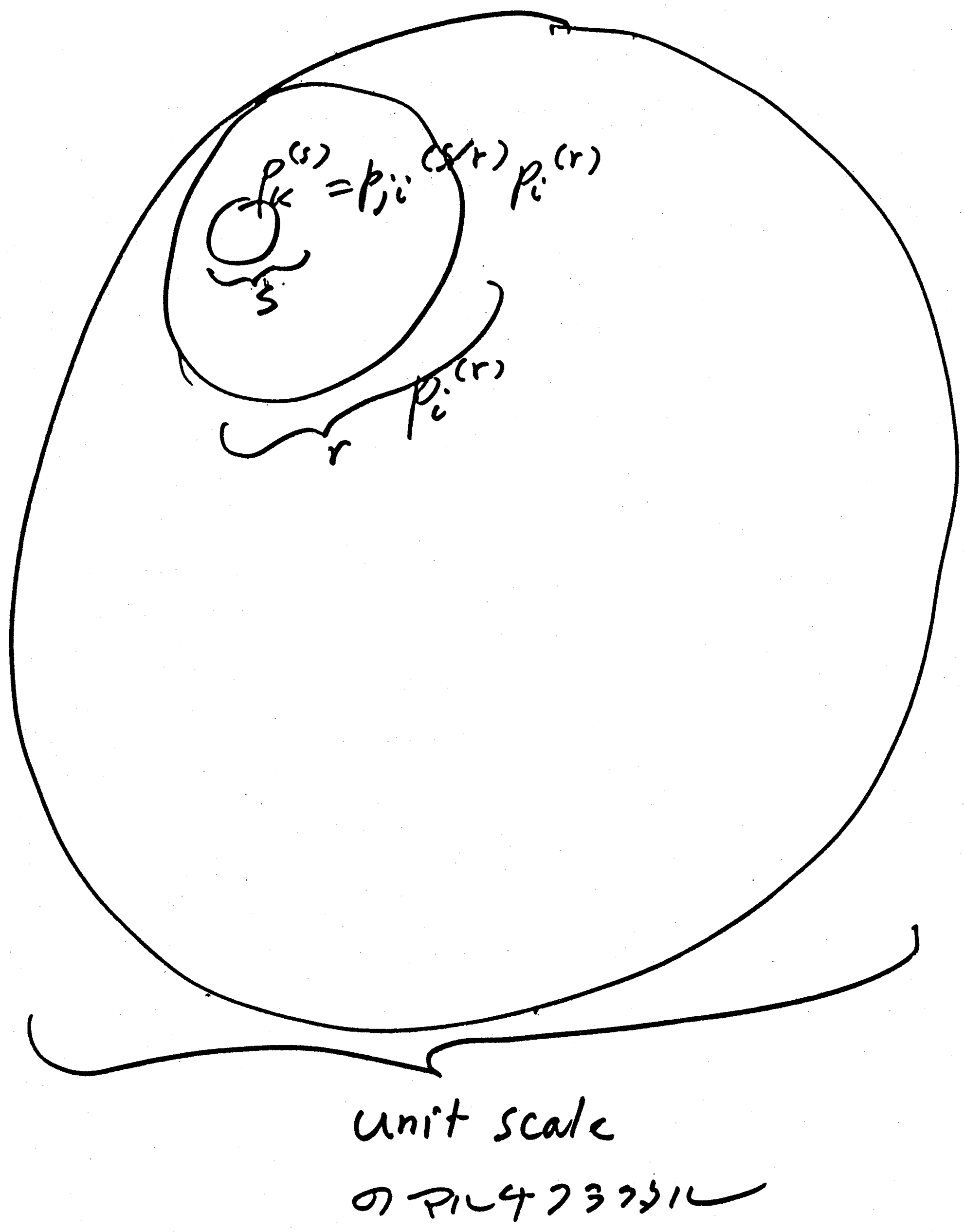


自己相似の構造
（全体のフラクタル構造は部分のフラクタル構造と同じ。）

Fig. 1